Skidding on a Lateral Inclined Road

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Summary
An earlier paper pointed out that the mathematical description of the skid process with lateral incline leads to a set of differential equations that can analytically be solved under simplifying conditions. A thorough elaboration shows that these simplifying conditions are fulfilled under normal circumstances but violated under conditions of low friction coefficient or steep lateral incline. This paper compares the analytical solution with numerical solutions under non-simplifying conditions.

Analytical deduction
According to Fig. 1 the skidding process of a body on a lateral incline may be described by the following system of differential equations

\[ \dot{x} = -\mu g \cos \alpha \cos \gamma \]

\[ \dot{y} = -\mu g \cos \alpha \sin \gamma + g \sin \alpha \]

\[ \tan \gamma = \frac{\dot{y}}{\dot{x}} \]

In [1] this system of non-linear differential equations of second order was uncoupled by the conditions

\[ \cos \gamma = 1 \quad ; \quad \sin \gamma = \tan \gamma \]

and was then solved analytically. In contrast to [1] we will use a dimensionless version of the differential equation system in this paper. Therefore we define the dimensionless variables.

\[ \xi = \frac{2}{v_0 T} x \quad ; \quad \eta = \frac{2 \mu}{\tan \alpha} \frac{2}{v_0 T} y \quad ; \quad \tau = \frac{t}{T} \]

with
With these definitions the differential equation system may be written as follows

\[ T = \frac{v_0}{\mu \ g \ \cos \alpha} \quad (6) \]

\[ \ddot{\xi} = -\frac{2}{\sqrt{1 + \left(\frac{\dot{\eta}}{\xi}\right)^2}} \quad (7) \]

\[ \ddot{\eta} = 4 + \ddot{\xi} \frac{\dot{\eta}}{\xi} \quad (8) \]

with

\[ \theta = \frac{\tan \alpha}{2\mu} \quad (9) \]

The linearisation according to eq. (4) is substituted by the condition

\[ \theta = 0 \quad (10) \]

Under this condition the analytical solution will look as follows

\[ \dot{\eta} = -4(1-\tau) \ln(1-\tau) \quad (11) \]

\[ \eta = 1-(1-\tau)^2 \left(1 - \ln(1-\tau)^2 \right) \quad (12) \]

respectively

\[ \eta' = -\ln(1-\xi) \quad (13) \]

\[ \eta = \xi + (1-\xi) \ln(1-\xi) \quad (14) \]

In [1] the admissibility of simplification eq. (4) respectively eq. (10) has been ‘proved’ with the analytical solution according to eq. (13). This proceeding is not quite correct, as this indeed is a necessary, but not a sufficient condition for the admissibility of simplification eq. (4). It could just as much be a self-fulfilling prophecy in that way that the solution suffices the
condition because this condition was a prerequisite of the analytical solution. So a really pre-

dictatory proof may only be performed by a numerical solution of the original system of dif-
ferential equations.

The dimensionless form of the system of differential equations according to eq. (7) clearly
illustrates that the permissibility of the simplification eq. (10) surly depends on the value of
parameter $\theta$. A value $\theta = \frac{1}{2}$ will yield in

$$\mu = \tan \alpha$$

respectively

$$\mu g \cos \alpha = g \sin \alpha$$

so that friction and propulsion by incline will balance one another, i. e. the skidding body will
end at a steady state where it moves downward with constant velocity. For even greater val-
ues of parameter $\theta$ the body will accelerare constantly while moving downward, for values $\theta$
$< \frac{1}{2}$ it will always come to rest.

**Fig. 2 – 4** show numerical solutions of the system of differential equations, for the following
values of parameter $\theta$: 0, 0.1, 0.2, 0.25 and 0.3. **Fig. 2** shows the dimensionless lateral veloc-
ity $\eta$ versus the dimensionless time $\tau$. In this diagram we may state deviation from the ana-
lytical solution for values $\tau > 0.5$. But as the velocity decreases constantly during the skidding
process, it will tend to zero at the end. Although a significant amount of time elapses while
the numerical solutions deviate from the analytical solution, not much distance is traversed
according to **Fig. 3**. For values $\xi < 0.9$ the lateral velocity $\eta$ hardly shows any difference in
contrast to the analytical solution.

Nevertheless **Fig. 4** illustrates that the lateral offset $\eta$ is affected significantly by a violation
of the assumption $\theta = 0$. For a value $\theta = 0.3$ the final lateral offset of the body is 1.5 times
greater than that of the analytical solution. If we take into account that the friction coefficient
of glazed frost may get as low as 0.05 – 0.15 [2], such values of parameter $\varphi$ may already be
reached at lateral inclines of only 3%. So the numerical solutions presented in this paper are
not just mind games.

In contrast to the solution presented in [1], the numerical solutions pre-
sented in this paper also give the lateral offset as a function of skid-
ing distance in longitudinal direc-
tion. It turns out that the most part
of the lateral offset is gained at the
very end of the skidding process.

The dimensionless presentation il-
lustrates furthermore that skidding
processes are always equal in
shape, independent of starting ve-
locity. The initial velocity and the
friction coefficient may therefore
just be considered as scale factors.

![Fig. 2: Dimensionless lateral velocity versus dimensionless time](image-url)
References

[1] Schimmelpfennig, K.-H.; Rennich, D.: 
Hinweise auf die Bedeutung der Fahrbahnneigung in der Unfallrekonstruktion. 
(Hints on the relevance of road camber to accident reconstruction) 
Verkehrsunfall und Fahrzeugtechnik 24 (1986), S. 221–223

Der Kraftschluß von Fahrzeugräder und Gummiproben auf vereister Oberfläche 
(The friction coefficient of vehicle tires and rubber specimen on glazed surface) 

In 1999 a new paper dealt with the problem, offering an analytical solution for the final offset in the general case $\theta > 0$. This paper is in English language:

[3] Searle, J.: 
Deviation of the Path of a Sliding Object due to Road Camber 
Verkehrsunfall und Fahrzeugtechnik 37 (1999), S. 41–42

This paper was originally published as

Hugemann, W.: 
Rutschvorgänge auf quergeneigter Fahrbahn. 
Verkehrsunfall und Fahrzeugtechnik 29 (1991), S. 101–102

In the original paper Eq. (8) looked like

$$\tilde{\eta} = 4 - \tilde{\xi} \frac{\gamma}{\xi}$$

and has been corrected here.

Fig. 3: Dimensionless lateral velocity versus dimensionless longitudinal distance

Fig. 4: Dimensionless lateral distance versus dimensionless longitudinal distance