Skidding on a Lateral Inclined Road

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Summary

An earlier paper pointed out that the mathematical description of the skid process with lateral incline leads to a set of differential equations that can analytically be solved under simplifying conditions. A thorough elaboration shows that these simplifying conditions are fulfilled under normal circumstances but violated under conditions of low friction coefficient or steep lateral incline. This paper compares the analytical solution with numerical solutions under non-simplifying conditions.

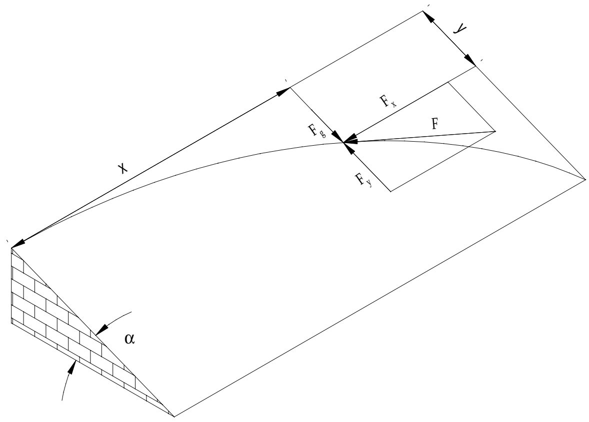

Fig. 1 Trajectory when skidding on an inclined plane

Analytical deduction

According to **Fig. 1** the skidding process of a body on a lateral incline may be described by the following system of differential equations

$$\ddot{x} = -\mu\, g \cos\alpha \cos\gamma \tag{1}$$

$$\ddot{y} = -\mu\, g \cos\alpha \sin\gamma + g \sin\alpha \tag{2}$$

$$\tan\gamma = \frac{\dot{y}}{\dot{x}} \tag{3}$$

In [1] this system of non-linear differential equations of second order was uncoupled by the conditions

$$\cos\gamma \approx 1 \quad ; \quad \sin\gamma \approx \tan\gamma \tag{4}$$

and was then solved analytically. In contrast to [1] we will use a dimensionless version of the differential equation system in this paper. Therefore we define the dimensionless variables.

$$\xi = \frac{2}{v_0 T}\, x \quad ; \quad \eta = \frac{2\mu}{\tan\alpha}\, \frac{2}{v_0 T}\, y \quad ; \quad \tau = \frac{t}{T} \tag{5}$$

with

$$\boldsymbol{T} = \frac{\boldsymbol{v_0}}{\mu\ \boldsymbol{g}\ \mathbf{cos}\,\alpha} \tag{6}$$

With these definitions the differential equation system may be written as follows

$$\ddot{\xi} = -\frac{\mathbf{2}}{\sqrt{\mathbf{1} + \left(\theta\,\frac{\dot{\eta}}{\dot{\xi}}\right)^2}} \tag{7}$$

$$\ddot{\eta} = \mathbf{4} + \ddot{\xi}\,\frac{\dot{\eta}}{\dot{\xi}} \tag{8}$$

with

$$\theta = \frac{\mathbf{tan}\,\alpha}{\boldsymbol{2}\mu} \tag{9}$$

The linearisation according to eq. (4) is substituted by the condition

$$\theta \approx \mathbf{0} \tag{10}$$

Under this condition the analytical solution will look as follows

$$\dot{\eta} = -\mathbf{4}\,(\mathbf{1}-\tau)\ \mathbf{ln}\ (\mathbf{1}-\tau) \tag{11}$$

$$\eta = \mathbf{1}-(\mathbf{1}-\tau)^2\ \left(\mathbf{1} - \mathbf{ln}\ (\mathbf{1}-\tau)^2\right) \tag{12}$$

respectively

$$\eta' = -\mathbf{ln}\ (\mathbf{1}-\xi) \tag{13}$$

$$\eta = \xi + (\mathbf{1}-\xi)\ \mathbf{ln}\ (\mathbf{1}-\xi) \tag{14}$$

In [1] the admissibility of simplification eq. (4) respectively eq. (10) has been ‘proved’ with the analytical solution according to eq. (13). This proceeding is not quite correct, as this indeed is a necessary, but not a sufficient condition for the admissibility of simplification eq. (4). It could just as much be a self-fulfilling prophecy in that way that the solution suffices the condition because this condition was a prerequisite of the analytical solution. So a really predicatory proof may only be performed by a numerical solution of the original system of differential equations.

The dimensionless form of the system of differential equations according to eq. (7) clearly illustrates that the permissibility of the simplification eq. (10) surly depends on the value of parameter θ. A value $\theta = ½$ will yield in

$$\mu = \tan\alpha \tag{15}$$

respectively

$$\mu\, g \cos\alpha = g \sin\alpha \tag{16}$$

so that friction and propulsion by incline will balance one another, i. e. the skidding body will end at a steady state where it moves downward with constant velocity. For even greater values of parameter θ the body will accellarate constantly while moving downward, for values $\theta < ½$ it will always come to rest.

Fig. 2 – 4 show numerical solutions of the system of differential equations, for the following values of parameter θ: 0, 0.1, 0.2, 0.25 and 0.3. **Fig. 2** shows the dimensionless lateral velocity $\dot{\eta}$ versus the dimensionless time τ. In this diagram we may state deviation from the analytical solution for values $\tau > 0.5$. But as the velocity decreases constantly during the skidding process, it will tend to zero at the end. Although a significant amount of time elapses while the numerical solutions deviate from the analytical solution, not much distance is traversed according to **Fig. 3**. For values $\xi < 0.9$ the lateral velocity $\dot{\eta}$ hardly shows any difference in contrast to the analytical solution.

Nevertheless **Fig. 4** illustrates that the lateral offset η is affected significantly by a violation of the assumption $\theta = 0$. For a value $\theta = 0.3$ the final lateral offset of the body is 1.5 times greater than that of the analytical solution. If we take into account that the friction coefficient of glazed frost may get as low as 0.05 – 0.15 [2], such values of parameter Θ may already be reached at lateral inclines of only 3%. So the numerical solutions presented in this paper are not just mind games.

In contrast to the solution presented in [1], the numerical solutions presented in this paper also give the lateral offset as a function of skidding distance in longitudinal direction. It turns out that the most part of the lateral offset is gained at the very end of the skidding process. The dimensionless presentation illustrates furthermore that skidding processes are always equal in shape, independent of starting velocity. The initial velocity and the friction coefficient may therefore just be considered as scale factors.

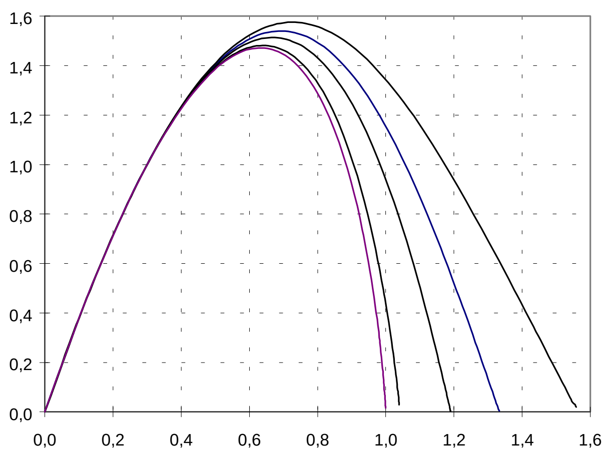

Fig. 2: Dimensionless lateral velocity versus dimensionless time

References

[1] Schimmelpfennig, K.-H.; Rennich, D.:
Hinweise auf die Bedeutung der Fahrbahnneigung in der Unfallrekonstruktion.
(Hints on the relevance of road camber to accident reconstruction)
Verkehrsunfall und Fahrzeugtechnik 24 (1986), S. 221–223

[2] Weber, R.:
Der Kraftschluß von Fahrzeugrädern und Gummiproben auf vereister Oberfläche
(The friction coefficient of vehicle tires and rubber specimen on glazed surface)
PhD Thesis, Karlsruhe 1970.

In 1999 a new paper dealt with the problem, offering an analytical solution for the final offset in the general case $\theta > 0$. This paper is in English language:

[3] Searle, J.:
Deviation of the Path of a Sliding Object due to Road Camber
Verkehrsunfall und Fahrzeugtechnik 37 (1999), S. 41–42

This paper was originally published as

Hugemann, W.:
Rutschvorgänge auf quergeneigter Fahrbahn.
Verkehrsunfall und Fahrzeugtechnik 29 (1991), S. 101–102

In the original paper Eq. (8) looked like

$$\ddot{\eta} = 4 - \ddot{\xi}\,\frac{\dot{\eta}}{\dot{\xi}}$$

and has been corrected here.

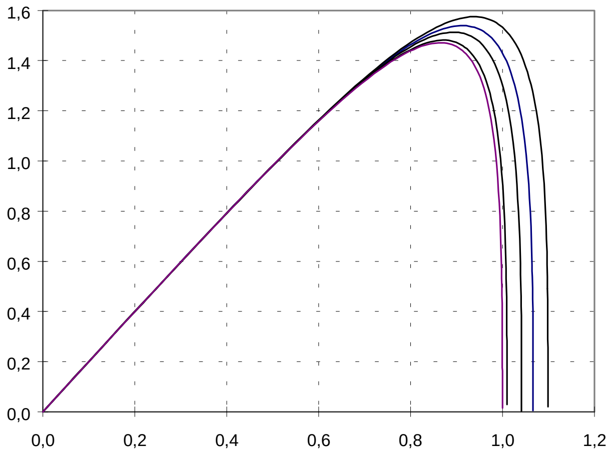

Fig. 3: Dimensionless lateral velocity versus dimensionless longitudinal distance

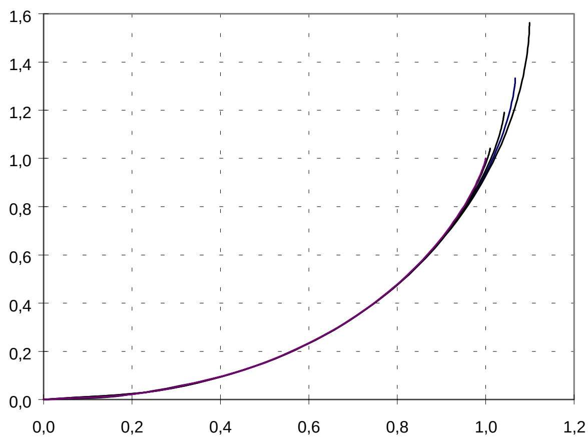

Fig. 4: Dimensionless lateral distance versus dimensionless longitudinal distance